

TRACES OF THE NEVANLINNA CLASS ON DISCRETE SEQUENCES

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ABSTRACT. We show that a discrete sequence Λ of the unit disk is the union of n interpolating sequences for the Nevanlinna class \mathcal{N} if and only if the trace of \mathcal{N} on Λ coincides with the space of functions on Λ for which the divided differences of order $n - 1$ are uniformly controlled by a positive harmonic function.

1. DEFINITIONS AND STATEMENT

This note deals with some properties of the classical *Nevanlinna class* consisting of the holomorphic functions in the unit disk \mathbb{D} for which $\log_+ |f|$ has a positive harmonic majorant. We denote by $\text{Har}_+(\mathbb{D})$ the set of non-negative harmonic functions in \mathbb{D} . Equivalently,

$$\mathcal{N} = \left\{ f \in \text{Hol}(\mathbb{D}) : \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta < \infty \right\}.$$

Definition. A discrete sequence of points Λ in \mathbb{D} is called *interpolating for \mathcal{N}* (denoted $\Lambda \in \text{Int } \mathcal{N}$) if the trace space $\mathcal{N}|_\Lambda$ is ideal, or equivalently, if for every $v \in \ell^\infty$ there exists $f \in \mathcal{N}$ such that

$$f(\lambda_n) = v_n, \quad n \in \mathbb{N}.$$

Interpolating sequences for the Nevanlinna class were first investigated by Naftalevitch [6]. A rather complete study was carried out much later in [4]. Let B denote the Blaschke product associated to a Blaschke sequence Λ . Let

$$b_\lambda(z) = \frac{z - \lambda}{1 - \bar{\lambda}z} \quad \text{and} \quad B_\lambda(z) = \frac{B(z)}{b_\lambda(z)}.$$

Let's also consider the pseudohyperbolic distance in \mathbb{D} , defined as

$$\rho(z, w) = \left| \frac{z - w}{1 - \bar{z}w} \right|,$$

and the corresponding pseudohyperbolic disks $D(z, r) = \{w \in \mathbb{D} : \rho(z, w) < r\}$.

According to [4, Theorem 1.2] $\Lambda \in \text{Int } \mathcal{N}$ if and only if there exists $H \in \text{Har}_+(\mathbb{D})$ such that

$$(1) \quad |B_\lambda(\lambda)| = (1 - |\lambda|)|B'(\lambda)| \geq e^{-H(\lambda)}, \quad \lambda \in \Lambda.$$

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Moreover in such case the trace space is

$$\mathcal{N}(\Lambda) = \{ \{ \omega(\lambda) \}_{\lambda \in \Lambda} : \exists H \in \text{Har}_+(\mathbb{D}), \log_+ |\omega(\lambda)| \leq H(\lambda), \lambda \in \Lambda \}.$$

Other properties and characterizations of Nevanlinna interpolating sequences have been given recently in [3]. In these terms $\Lambda \in \text{Int } \mathcal{N}$ when for every sequence $\omega(\Lambda) \in \mathcal{N}(\Lambda)$ there exists $f \in \mathcal{N}$ such that $f(\lambda) = \omega(\lambda)$, $\lambda \in \Lambda$. In terms of the restriction operator

$$\begin{aligned} \mathcal{R}_\Lambda : \mathcal{N} &\longrightarrow \mathcal{N}(\Lambda) \\ f &\mapsto \{f(\lambda)\}_{\lambda \in \Lambda}, \end{aligned}$$

Λ is interpolating when $\mathcal{R}_\Lambda(\mathcal{N}) = \mathcal{N}(\Lambda)$.

Definition 1.1. Let Λ be a discrete sequence in \mathbb{D} and ω a function given on Λ . The *pseudohyperbolic divided differences* of ω are defined by induction as follows

$$\begin{aligned} \Delta^0 \omega(\lambda_1) &= \omega(\lambda_1), \\ \Delta^j \omega(\lambda_1, \dots, \lambda_{j+1}) &= \frac{\Delta^{j-1} \omega(\lambda_2, \dots, \lambda_{j+1}) - \Delta^{j-1} \omega(\lambda_1, \dots, \lambda_j)}{b_{\lambda_1}(\lambda_{j+1})} \quad j \geq 1. \end{aligned}$$

For any $n \in \mathbb{N}$, denote

$$\Lambda^n = \{ (\lambda_1, \dots, \lambda_n) \in \Lambda \times \overset{n}{\cdots} \times \Lambda : \lambda_j \neq \lambda_k \text{ if } j \neq k \},$$

and consider the set $X^{n-1}(\Lambda)$ consisting of the functions defined in Λ with divided differences of order $n-1$ uniformly controlled by a positive harmonic function H i.e., such that for some $H \in \text{Har}_+(\mathbb{D})$,

$$\sup_{(\lambda_1, \dots, \lambda_n) \in \Lambda^n} |\Delta^{n-1} \omega(\lambda_1, \dots, \lambda_n)| e^{-[H(\lambda_1) + \dots + H(\lambda_n)]} < +\infty.$$

Lemma 1.2. Let $n \in \mathbb{N}$. For any sequence $\Lambda \subset \mathbb{D}$, we have $X^n(\Lambda) \subset X^{n-1}(\Lambda) \subset \dots \subset X^0(\Lambda) = \mathcal{N}(\Lambda)$.

Proof. Assume that $\omega(\Lambda) \in X^n(\Lambda)$, that is,

$$\sup_{(\lambda_1, \dots, \lambda_{n+1}) \in \Lambda^{n+1}} \left| \frac{\Delta^{n-1} \omega(\lambda_2, \dots, \lambda_{n+1}) - \Delta^{n-1} \omega(\lambda_1, \dots, \lambda_n)}{b_{\lambda_1}(\lambda_{n+1})} \right| e^{-[H(\lambda_1) + \dots + H(\lambda_{n+1})]} < \infty.$$

Then, given $(\lambda_1, \dots, \lambda_n) \in \Lambda^n$ and taking $\lambda_1^0, \dots, \lambda_n^0$ from a finite set (for instance the n first $\lambda_j^0 \in \Lambda$ different of all λ_j) we have

$$\begin{aligned} \Delta^{n-1} \omega(\lambda_1, \dots, \lambda_n) &= \frac{\Delta^{n-1} \omega(\lambda_1, \dots, \lambda_n) - \Delta^{n-1} \omega(\lambda_1^0, \lambda_1, \dots, \lambda_{n-1})}{b_{\lambda_1^0}(\lambda_n)} b_{\lambda_1^0}(\lambda_n) + \\ &+ \frac{\Delta^{n-1} \omega(\lambda_1^0, \lambda_1, \dots, \lambda_{n-1}) - \Delta^{n-1} \omega(\lambda_2^0, \lambda_1^0, \dots, \lambda_{n-2})}{b_{\lambda_2^0}(\lambda_{n-1})} b_{\lambda_2^0}(\lambda_{n-1}) + \dots + \\ &+ \frac{\Delta^{n-1} \omega(\lambda_{n-1}^0, \dots, \lambda_1^0, \lambda_1) - \Delta^{n-1} \omega(\lambda_n^0, \dots, \lambda_1^0)}{b_{\lambda_n^0}(\lambda_1)} b_{\lambda_n^0}(\lambda_1) + \Delta^{n-1} \omega(\lambda_n^0, \dots, \lambda_1^0) \end{aligned}$$

Since $\omega \in X^{n-1}(\Lambda)$ there exists $H \in \text{Har}_+(\mathbb{D})$ and a constant $K(\lambda_1^0, \dots, \lambda_n^0)$ such that

$$\begin{aligned} |\Delta^{n-1}\omega(\lambda_1, \dots, \lambda_n)| &\leq e^{H(\lambda_1^0)+H(\lambda_1)\dots+H(\lambda_n)}\rho(\lambda_1^0, \lambda_n) + e^{H(\lambda_1^0)+H(\lambda_2^0)\dots+H(\lambda_{n-1})}\rho(\lambda_2^0, \lambda_{n-1}) + \\ &\quad + \dots + e^{H(\lambda_1^0)+\dots+H(\lambda_n^0)+H(\lambda_1)}\rho(\lambda_n^0, \lambda_1) + \Delta^{n-1}\omega(\lambda_n^0, \dots, \lambda_1^0) \\ &\leq K(\lambda_1^0, \dots, \lambda_n^0) e^{H(\lambda_1)+\dots+H(\lambda_n)}, \end{aligned}$$

and the statement follows. \blacksquare

The main result of this note is modelled after Vasyunin's description of the sequences Λ in \mathbb{D} such that the trace of the algebra of bounded holomorphic functions H^∞ on Λ equals the space of pseudohyperbolic divided differences of order n (see [7], [8]). Similar results hold also for Hardy spaces (see [1] and [2]) and the Hörmander algebras, both in \mathbb{C} and in \mathbb{D} [5]. The analogue in our context is the following.

Main Theorem. *The identity $\mathcal{N}|\Lambda = X^{n-1}(\Lambda)$ holds if and only if Λ is the union of n interpolating sequences for \mathcal{N} .*

2. GENERAL PROPERTIES

Throughout the proofs we will use repeatedly the well-known *Harnack inequalities*: for $H \in \text{Har}_+(\mathbb{D})$ and $z, w \in \mathbb{D}$,

$$\frac{1 - \rho(z, w)}{1 + \rho(z, w)} \leq \frac{H(z)}{H(w)} \leq \frac{1 + \rho(z, w)}{1 - \rho(z, w)}.$$

We shall always assume, without loss of generality, that $H \in \text{Har}_+(\mathbb{D})$ is big enough so that for $z \in D(\lambda, e^{-H(\lambda)})$ the inequalities $1/2 \leq H(z)/H(\lambda) \leq 2$ hold. Actually it is sufficient to assume $\inf\{H(z) : z \in \mathbb{D}\} \geq \log 3$.

We begin by showing that one of the inclusions of the Main Theorem is immediate.

Proposition 2.1. *For all $n \in \mathbb{N}$, the inclusion $\mathcal{N}|\Lambda \subset X^{n-1}(\Lambda)$ holds.*

Proof. Let $f \in \mathcal{N}$. Let us show by induction on $j \geq 1$ that there exists $H \in \text{Har}_+(\mathbb{D})$ such that

$$|\Delta^{j-1}f(z_1, \dots, z_j)| \leq e^{H(z_1)+\dots+H(z_j)} \quad \text{for all } (z_1, \dots, z_j) \in \mathbb{D}^j.$$

As $f \in \mathcal{N}$, there exists $H \in \text{Har}_+(\mathbb{D})$ such that $|\Delta^0 f(z_1)| = |f(z_1)| \leq e^{H(z_1)}$, $z_1 \in \mathbb{D}$.

Assume that the property is true for j and let $(z_1, \dots, z_{j+1}) \in \mathbb{D}^{j+1}$. Fix z_1, \dots, z_j and consider z_{j+1} as the variable in the function

$$\Delta^j f(z_1, \dots, z_{j+1}) = \frac{\Delta^{j-1}f(z_2, \dots, z_{j+1}) - \Delta^{j-1}f(z_1, \dots, z_j)}{b_{z_1}(z_{j+1})}.$$

By the induction hypothesis, there exists $H \in \text{Har}_+(\mathbb{D})$ such that

$$|\Delta^j f(z_1, \dots, z_{j+1})| \leq \frac{1}{\rho(z_1, z_{j+1})} (e^{H(z_2)+\dots+H(z_{j+1})} + e^{H(z_1)+\dots+H(z_j)}).$$

If $\rho(z_1, z_{j+1}) \geq 1/2$ we get directly

$$|\Delta^j f(z_1, \dots, z_{j+1})| \leq 4e^{H(z_1)+\dots+H(z_{j+1})},$$

and choosing for instance $\tilde{H} = H + \log 4$ we get the desired estimate.

If $\rho(z_1, z_{j+1}) \leq 1/2$ we apply the maximum principle and Harnack's inequalities

$$\begin{aligned} |\Delta^j f(z_1, \dots, z_{j+1})| &\leq \sup_{\xi: \rho(\xi, z_{j+1})=1/2} |\Delta^j f(z_1, \dots, z_j, \xi_{j+1})| \\ &\leq \sup_{\xi: \rho(\xi, z_{j+1})=1/2} 4e^{H(z_1)+\dots+H(z_j)+H(\xi)} \\ &\leq 4e^{2[H(z_1)+\dots+H(z_j)+H(z_{j+1})]}. \end{aligned}$$

Choosing here $\tilde{H} = 2H + \log 4$ we get the desired estimate. ■

Definition 2.2. A sequence Λ is *weakly separated* if there exists $H \in \text{Har}_+(\mathbb{D})$ such that the disks $D(\lambda, e^{-H(\lambda)})$, $\lambda \in \Lambda$, are pairwise disjoint.

Remark 2.3. If Λ is weakly separated then $X^0(\Lambda) = X^n(\Lambda)$, for all $n \in \mathbb{N}$.

By Lemma 1.2, to see this it is enough to prove (by induction) that $X^0(\Lambda) \subset X^n(\Lambda)$ for all $n \in \mathbb{N}$.

For $n = 0$ this is trivial.

Assume now that $X^0(\Lambda) \subset X^{n-1}(\Lambda)$ and take $\omega(\Lambda) \in X^0(\Lambda)$. Since $\rho(\lambda_1, \lambda_{n+1}) \geq e^{-H_0(\lambda_1)}$ for some $H_0 \in \text{Har}_+(\mathbb{D})$ we have

$$\begin{aligned} |\Delta^n \omega(\lambda_1, \dots, \lambda_{n+1})| &= \left| \frac{\Delta^{n-1} \omega(\lambda_2, \dots, \lambda_{n+1}) - \Delta^{n-1} \omega(\lambda_1, \dots, \lambda_n)}{b_{\lambda_1}(\lambda_{n+1})} \right| \\ &\leq e^{H_0(\lambda_1)} (e^{H(\lambda_2)+\dots+H(\lambda_{n+1})} + e^{H(\lambda_1)+\dots+H(\lambda_n)}) \end{aligned}$$

for some $H \in \text{Har}_+(\mathbb{D})$. Taking $\tilde{H} = H + H_0$ we are done.

Lemma 2.4. Let $n \geq 1$. The following assertions are equivalent:

- (a) Λ is the union of n weakly separated sequences,
- (b) There exist $H \in \text{Har}_+(\mathbb{D})$ such that

$$\sup_{\lambda \in \Lambda} \#[\Lambda \cap D(\lambda, e^{-H(\lambda)})] \leq n.$$

- (c) $X^{n-1}(\Lambda) = X^n(\Lambda)$.

Proof. (a) \Rightarrow (b). This is clear, by the weak separation.

(b) \Rightarrow (a). We proceed by induction on $j = 1, \dots, n$. For $j = 1$, it is again clear by the definition of weak separation. Assume the property true for $j-1$. Let $H \in \text{Har}_+(\mathbb{D})$, $\inf\{H(z) : z \in \mathbb{D}\} \geq \log 3$, be such that $\sup_{\lambda \in \Lambda} \#[\Lambda \cap D(\lambda, e^{-H(\lambda)})] \leq j$. We split the sequence $\Lambda = \Lambda_a \cup \Lambda_b$ where

$$\begin{aligned} \Lambda_a &= \bigcup_{\{\lambda \in \Lambda : \#(\Lambda \cap D(\lambda, e^{-10H(\lambda)})) = j\}} (\Lambda \cap D(\lambda, e^{-10H(\lambda)})) \\ \Lambda_b &= \Lambda \setminus \Lambda_a \end{aligned}$$

Now, for every $\lambda \in \Lambda_b$, we have $\#(\Lambda \cap D(\lambda, e^{-10H(\lambda)})) \leq j-1$, and by the induction hypothesis, Λ_b splits into $j-1$ separated sequences $\Lambda_1, \dots, \Lambda_{j-1}$.

In the case $\lambda \in \Lambda_a$, there is obviously no point in the annulus $D(\lambda, e^{-H(\lambda)}) \setminus D(\lambda, e^{-10H(\lambda)})$ which means that the j points in $D(\lambda, e^{-10H(\lambda)})$ are far from the other points of Λ . So we can add each one of these j points in a weakly separated way to one of the sequences $\Lambda_1, \dots, \Lambda_{j-1}$, and the j -th point in a new sequence Λ_j (which is of course weakly separated since the groups $\Lambda \cap D(\lambda, e^{-10H(\lambda)})$ appearing in Λ_a are weakly separated).

(b) \Rightarrow (c). It remains to see that $X^{n-1}(\Lambda) \subset X^n(\Lambda)$. Given $\omega(\Lambda) \in X^{n-1}(\Lambda)$ and points $(\lambda_1, \dots, \lambda_{n+1}) \in \Lambda^{n+1}$, we have to estimate $\Delta^n \omega(\lambda_1, \dots, \lambda_{n+1})$. Under the assumption (b), at least one of these $n+1$ points is not in the disk $D(\lambda_1, e^{-H(\lambda_1)})$. Note that Λ^n is invariant by permutation of the $n+1$ points, thus we may assume that $\rho(\lambda_1, \lambda_{n+1}) \geq e^{-H(\lambda_1)}$. Using the fact that $\omega(\Lambda) \in X^{n-1}(\Lambda)$, there exists $H_0 \in \text{Har}_+(\mathbb{D})$ such that

$$\begin{aligned} |\Delta^n \omega(\lambda_1, \dots, \lambda_{n+1})| &\leq \frac{|\Delta^{n-1} \omega(\lambda_2, \dots, \lambda_{n+1})| + |\Delta^{n-1} \omega(\lambda_1, \dots, \lambda_n)|}{\rho(\lambda_1, \lambda_{n+1})} \\ &\leq e^{H(\lambda_1)} (e^{H_0(\lambda_2) + \dots + H_0(\lambda_{n+1})} + e^{H_0(\lambda_1) + \dots + H_0(\lambda_n)}) \\ &\leq 2e^{H(\lambda_1)} e^{H_0(\lambda_1) + \dots + H_0(\lambda_{n+1})}. \end{aligned}$$

Taking $\tilde{H} = H_0 + H + \log 2$ we get the desired estimate.

(c) \Rightarrow (b). We prove this by contraposition. Assume that for all $H \in \text{Har}_+(\mathbb{D})$ there exists $\lambda \in \Lambda$ such that

$$(2) \quad \#[\Lambda \cap D(\lambda, e^{-H(\lambda)})] > n.$$

Consider the partition of \mathbb{D} into the dyadic squares

$$Q_{k,j} = \{z = re^{i\theta} \in \mathbb{D} : 1 - 2^{-k} \leq r < 1 - 2^{-k-1}, j \frac{2\pi}{k} \leq \theta < (j+1) \frac{2\pi}{k}\},$$

where $k \geq 0$ and $j = 0, \dots, 2^k - 1$.

Let $\Lambda_{k,j} = \Lambda \cap Q_{k,j}$ and

$$r_{k,j} = \inf\{r > 0 : \exists \lambda \in \Lambda_{k,j} : \#(\Lambda \cap \overline{D(\lambda, r)}) \geq n+1\}.$$

Take $\alpha_{k,j} \in \Lambda_{k,j}$ such that $\#(\Lambda \cap \overline{D(\alpha_{k,j}, r_{k,j})}) \geq n+1$.

Claim: For all $H \in \text{Har}_+(\mathbb{D})$,

$$\inf_{k,j} \frac{r_{k,j}}{e^{-H(\alpha_{k,j})}} = 0.$$

To see this assume otherwise that there exist $H \in \text{Har}_+(\mathbb{D})$ and $\eta > 0$ with

$$\frac{r_{k,j}}{e^{-H(\alpha_{k,j})}} \geq \eta.$$

In particular, by Harnack's inequalities,

$$(3) \quad \log \frac{1}{r_{k,j}} \leq 3H(z) + \log\left(\frac{1}{\eta}\right), \quad z \in Q_{k,j}.$$

Let $\tilde{H} := \log(2/\eta) + 4H \in \text{Har}_+(\mathbb{D})$. By the hypothesis (2) there exist $k_0 \geq 0$, $j_0 \in \{0, \dots, 2^{k_0} - 1\}$, $\lambda_{k_0, j_0} \in \Lambda_{k_0, j_0}$ such that

$$\# \left[\Lambda \cap \overline{D(\lambda_{k_0, j_0}, e^{-\tilde{H}(\lambda_{k_0, j_0})})} \right] \geq n + 1.$$

In particular, by definition of $r_{k, j}$, we have $r_{k_0, j_0} \leq e^{-\tilde{H}(\lambda_{k_0, j_0})}$, that is

$$\log \frac{1}{r_{k_0, j_0}} \geq \tilde{H}(\lambda_{k_0, j_0}) = \log\left(\frac{2}{\eta}\right) + 4H(\lambda_{k_0, j_0}),$$

which contradicts (3).

Now take a separated sequence $\mathcal{L} \subset \{\alpha_{k, j}\}_{k, j}$ for which the disks $D(\alpha, r_\alpha)$, $\alpha \in \mathcal{L}$, are disjoint, where for $\alpha = \alpha_{k, j} \in \mathcal{L}$ we denote $r_\alpha = r_{k, j}$. Given $\alpha \in \mathcal{L}$, let $\lambda_1^\alpha, \dots, \lambda_n^\alpha$ be its n nearest (not necessarily unique) points, arranged by increasing distance. Notice that $\rho(\alpha, \lambda_n^\alpha) = r_\alpha$.

In order to construct a sequence $\omega(\Lambda) \in X^{n-1}(\Lambda) \setminus X^n(\Lambda)$, put

$$\begin{cases} \omega(\alpha) = \prod_{j=1}^{n-1} b_\alpha(\lambda_j^\alpha), & \text{for all } \alpha \in \mathcal{L} \\ \omega(\lambda) = 0 & \text{if } \lambda \in \Lambda \setminus \mathcal{L}. \end{cases}$$

To see that $\omega(\Lambda) \in X^{n-1}(\Lambda)$ let us estimate $\Delta^{n-1}\omega(\lambda_1, \dots, \lambda_n)$ for any given $(\lambda_1, \dots, \lambda_n) \in \Lambda^n$. By the separation conditions on \mathcal{L} , we know that none of the λ_j^α is in \mathcal{L} . Hence, we may assume that at most one of the points is in \mathcal{L} . On the other hand, it is clear that $\Delta^{n-1}\omega(\lambda_1, \dots, \lambda_n) = 0$ if all the points are in $\Lambda \setminus \mathcal{L}$. Thus, taking into account that Δ^{n-1} is invariant by permutations, we will only consider the case where λ_n is some $\alpha \in \mathcal{L}$ and $\lambda_1, \dots, \lambda_{n-1}$ are in $\Lambda \setminus \mathcal{L}$. In that case,

$$|\Delta^{n-1}\omega(\lambda_1, \dots, \lambda_{n-1}, \alpha)| = |\omega(\alpha)| \prod_{j=1}^{n-1} \rho(\alpha, \lambda_j)^{-1} = \prod_{j=1}^{n-1} \frac{\rho(\alpha, \lambda_j^\alpha)}{\rho(\alpha, \lambda_j)} \leq 1,$$

as desired.

On the other hand, a similar computation yields

$$|\Delta^n \omega(\lambda_1^\alpha, \dots, \lambda_n^\alpha, \alpha)| = |\omega(\alpha)| \prod_{j=1}^n \rho(\alpha, \lambda_j^\alpha)^{-1} = \rho(\alpha, \lambda_n^\alpha)^{-1} = r_\alpha^{-1}.$$

The Claim above prevents the existence of $H \in \text{Har}_+(\mathbb{D})$ such that

$$r_\alpha^{-1} = |\Delta^n \omega(\lambda_1^\alpha, \dots, \lambda_n^\alpha, \alpha)| e^{-(H(\lambda_1^\alpha) + \dots + H(\lambda_n^\alpha) + H(\alpha))} \leq C,$$

since otherwise, again by Harnack's inequalities, we would have

$$r_\alpha^{-1} \leq e^{3(n+1)H(\alpha)}, \quad \alpha \in \mathcal{L}.$$

■

It is clear from the characterization (1) of interpolating sequences for \mathcal{N} that such sequences must be weakly separated. The previous result gives another way of showing it.

Corollary 2.5. *If Λ is an interpolating sequence, then it is weakly separated.*

Proof. If Λ is an interpolating sequence, then $\mathcal{N}|\Lambda = X^0(\Lambda)$. On the other hand, by Proposition 2.1, $\mathcal{N}|\Lambda \subset X^1(\Lambda)$. Thus $X^0(\Lambda) = X^1(\Lambda)$. We conclude by the preceding lemma applied to the particular case $n = 1$. \blacksquare

The covering provided by the following result will be useful.

Lemma 2.6. *Let $\Lambda_1, \dots, \Lambda_n$ be weakly separated sequences. There exist $H \in \text{Har}_+(\mathbb{D})$, positive constants α, β , a subsequence $\mathcal{L} \subset \Lambda_1 \cup \dots \cup \Lambda_n$ and disks $D_\lambda = D(\lambda, r_\lambda)$, $\lambda \in \mathcal{L}$, such that*

- (i) $\Lambda_1 \cup \dots \cup \Lambda_n \subset \bigcup_{\lambda \in \mathcal{L}} D_\lambda$,
- (ii) $e^{-\beta H(\lambda)} \leq r_\lambda \leq e^{-\alpha H(\lambda)}$ for all $\lambda \in \mathcal{L}$,
- (iii) $\rho(D_\lambda, D_{\lambda'}) \geq e^{-\beta H(\lambda)}$ for all $\lambda, \lambda' \in \mathcal{L}$, $\lambda \neq \lambda'$.
- (iv) $\#(\Lambda_j \cap D_\lambda) \leq 1$ for all $j = 1, \dots, n$ and $\lambda \in \mathcal{L}$.

Proof. Let $H \in \text{Har}_+(\mathbb{D})$ be such that

$$(4) \quad \rho(\lambda, \lambda') \geq e^{-H(\lambda)}, \quad \forall \lambda, \lambda' \in \Lambda_j, \lambda \neq \lambda', \forall j = 1, \dots, n.$$

We will proceed by induction on $k = 1, \dots, n$ to show the existence of a subsequence $\mathcal{L}_k \subset \Lambda_1 \cup \dots \cup \Lambda_k$ such that:

- (i)_k $\Lambda_1 \cup \dots \cup \Lambda_k \subset \bigcup_{\lambda \in \mathcal{L}_k} D(\lambda, R_\lambda^k)$,
- (ii)_k $e^{-\beta_k H(\lambda)} \leq R_\lambda^k \leq e^{-\alpha_k H(\lambda)}$,
- (iii)_k $\rho(D(\lambda, R_\lambda^k), D(\lambda', R_{\lambda'}^k)) \geq e^{-\beta_k H(\lambda)}$ for any $\lambda, \lambda' \in \mathcal{L}_k$, $\lambda \neq \lambda'$.

Then it suffices to chose $\mathcal{L} = \mathcal{L}_n$, $\alpha = \alpha_n$, $\beta = \beta_n$, $r_\lambda = R_\lambda^n$. The weak separation and the fact that $r_\lambda < e^{-H(\lambda)}/3$ implies that $\#\Lambda_j \cap D(\lambda, r_\lambda) \leq 1$, $j = 1, \dots, k$, hence the lemma follows.

For $k = 1$, the property is clearly verified with $\mathcal{L}_1 = \Lambda_1$ and $R_\lambda^1 = e^{-CH(\lambda)}$, with C big enough so that (iii)₁ holds ($C = 3$, for instance). Properties (i)₁, (ii)₁ follow immediately.

Assume the property true for k and split $\mathcal{L}_k = \mathcal{M}_1 \cup \mathcal{M}_2$ and $\Lambda_{k+1} = \mathcal{N}_1 \cup \mathcal{N}_2$, where

$$\begin{aligned} \mathcal{M}_1 &= \{\lambda \in \mathcal{L}_k : D(\lambda, R_\lambda^k + 1/4 e^{-\beta_k H(\lambda)}) \cap \Lambda_{k+1} \neq \emptyset\}, \\ \mathcal{N}_1 &= \Lambda_{k+1} \cap \bigcup_{\lambda \in \mathcal{L}_k} D(\lambda, R_\lambda^k + 1/4 e^{-\beta_k H(\lambda)}), \\ \mathcal{M}_2 &= \mathcal{L}_k \setminus \mathcal{M}_1, \\ \mathcal{N}_2 &= \Lambda_{k+1} \setminus \mathcal{N}_1. \end{aligned}$$

Now, we put $\mathcal{L}_{k+1} = \mathcal{L}_k \cup \mathcal{N}_2$ and define the radii R_λ^{k+1} as follows:

$$R_\lambda^{k+1} = \begin{cases} R_\lambda^k + 1/4 e^{-\beta_k H(\lambda)} & \text{if } \lambda \in \mathcal{M}_1, \\ R_\lambda^k & \text{if } \lambda \in \mathcal{M}_2, \\ 1/8 e^{-\beta_k H(\lambda)} & \text{if } \lambda \in \mathcal{N}_2. \end{cases}$$

It is clear that (i)_{k+1} holds:

$$\Lambda_1 \cup \dots \cup \Lambda_{k+1} \subset \bigcup_{\lambda \in \mathcal{L}_{k+1}} D(\lambda, R_\lambda^{k+1}).$$

Also, by the induction hypothesis,

$$\frac{1}{8}e^{-\beta_k H(\lambda)} \leq R_\lambda^{k+1} \leq e^{-\alpha_k H(\lambda)} + \frac{1}{4}e^{-\beta_k H(\lambda)}.$$

Thus, to see $(ii)_{k+1}$ there is enough to choose $\alpha_{k+1}, \beta_{k+1}$ such that

$$e^{-\alpha_k H(\lambda)} + \frac{1}{4}e^{-\beta_k H(\lambda)} \leq e^{-\alpha_{k+1} H(\lambda)},$$

for instance $\alpha_{k+1} = \alpha_k - 1$, and

$$(5) \quad \frac{1}{8}e^{-\beta_k H(\lambda)} \geq e^{-\beta_{k+1} H(\lambda)},$$

that is $\beta_{k+1} H(\lambda) \geq \beta_k H(\lambda) + \log 8$. Assuming without loss of generality that $H(\lambda) \geq \log 8$, there is enough choosing $\beta_{k+1} \geq \beta_k + 1$.

In order to prove $(iii)_k$ take now $\lambda, \lambda' \in \mathcal{L}_{k+1}$, $\lambda \neq \lambda'$. Notice that

$$\rho(D(\lambda, R_\lambda^{k+1}), D(\lambda', R_{\lambda'}^{k+1})) = \rho(\lambda, \lambda') - R_\lambda^{k+1} - R_{\lambda'}^{k+1}.$$

Split into four different cases:

1. $\lambda, \lambda' \in \mathcal{L}_k$. Assume without loss of generality that $H(\lambda) \leq H(\lambda')$. Then, by the definition of R_λ^{k+1} , we see that

$$\rho(D(\lambda, R_\lambda^{k+1}), D(\lambda', R_{\lambda'}^{k+1})) = \rho(\lambda, \lambda') - R_\lambda^k - R_{\lambda'}^k - \frac{1}{4}e^{-\beta_k H(\lambda)} - \frac{1}{4}e^{-\beta_k H(\lambda')}.$$

By inductive hypothesis

$$\rho(\lambda, \lambda') - R_\lambda^k - R_{\lambda'}^k = \rho(D(\lambda, R_\lambda^k), D(\lambda', R_{\lambda'}^k)) \geq e^{-\beta_k H(\lambda)}.$$

Thus, by (5),

$$\rho(D(\lambda, R_\lambda^{k+1}), D(\lambda', R_{\lambda'}^{k+1})) \geq e^{-\beta_k H(\lambda)} - \frac{1}{2}e^{-\beta_k H(\lambda)} = \frac{1}{2}e^{-\beta_k H(\lambda)} \geq e^{-\beta_{k+1} H(\lambda)}.$$

2. $\lambda, \lambda' \in \mathcal{N}_2$. Assume also $H(\lambda) \leq H(\lambda')$. Condition (4) implies $\rho(\lambda, \lambda') \geq e^{-H(\lambda)}$, hence

$$\rho(D(\lambda, R_\lambda^{k+1}), D(\lambda', R_{\lambda'}^{k+1})) \geq e^{-H(\lambda)} - \frac{1}{4}e^{-\beta_k H(\lambda)}.$$

If $\beta_k \geq 2$, by (5) we have

$$\rho(D(\lambda, R_\lambda^{k+1}), D(\lambda', R_{\lambda'}^{k+1})) \geq e^{-2H(\lambda)} \geq e^{-\beta_k H(\lambda)} \geq e^{-\beta_{k+1} H(\lambda)}.$$

3. $\lambda \in \mathcal{M}_1, \lambda' \in \mathcal{N}_2$ By definition of \mathcal{M}_1 there exists $\beta \in \mathcal{N}_1$ such that

$$\rho(\lambda, \beta) \leq R_\lambda^k + \frac{1}{4}e^{-\beta_k H(\lambda)}.$$

Then, using (4) on $\beta, \lambda' \in \Lambda_{k+1}$, we have, by Harnack's inequalities (if $\beta_k \geq 4$),

$$\begin{aligned} \rho(\lambda, \lambda') &\geq \rho(\beta, \lambda') - \rho(\lambda, \beta) \geq e^{-H(\beta)} - R_\lambda^k - \frac{1}{4}e^{-\beta_k H(\lambda)} \geq e^{-2H(\lambda)} - \frac{5}{4}e^{-\beta_k H(\lambda)} \\ &\geq e^{-4H(\lambda)} \geq e^{-\beta_k H(\lambda)} \geq e^{-\beta_{k+1} H(\lambda)}. \end{aligned}$$

4. $\lambda \in \mathcal{M}_2, \lambda' \in \mathcal{N}_2$. Taking into account the definition of $R_\lambda^{k+1}, R_{\lambda'}^{k+1}$ we have

$$\rho(D(\lambda, R_\lambda^{k+1}), D(\lambda', R_{\lambda'}^{k+1})) = \rho(\lambda, \lambda') - R_\lambda^k - \frac{1}{8}e^{-\beta_k H(\lambda)}$$

Since

$$\rho(\lambda, \lambda') - R_\lambda^k \geq \rho(D(\lambda, R_\lambda^k), D(\lambda', R_{\lambda'}^k)),$$

by inductive hypothesis and by (5)

$$\rho(D(\lambda, R_\lambda^{k+1}), D(\lambda', R_{\lambda'}^{k+1})) \geq \frac{1}{4}e^{-\beta_k H(\lambda)} - \frac{1}{8}e^{-\beta_k H(\lambda)} \geq e^{-\beta_{k+1} H(\lambda)}.$$

All together, it is enough to start with $C > n$, define $\alpha_1 = \beta_1 = C$, and then define α_k, β_k inductively by

$$\alpha_{k+1} = \alpha_k - 1 = \dots = C - k, \quad \beta_{k+1} = \beta_k + 1 = \dots = C + k.$$

■

3. PROOF OF MAIN THEOREM. NECESSITY

Assume $\mathcal{N}|\Lambda = X^{n-1}(\Lambda)$, $n \geq 2$. Using Proposition 2.1, we have $X^{n-1}(\Lambda) = X^n(\Lambda)$, and by Lemma 2.4 we deduce that $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_n$, where $\Lambda_1, \dots, \Lambda_n$ are weakly separated sequences. We want to show that each Λ_j is an interpolating sequence.

Let $\omega(\Lambda_j) \in \mathcal{N}(\Lambda_j) = X^0(\Lambda_j)$. Let $\cup_{\lambda \in \mathcal{L}} D_\lambda$ be the covering of Λ given by Lemma 2.6. We extend $\omega(\Lambda_j)$ to a sequence $\omega(\Lambda)$ which is constant on each $D_\lambda \cap \Lambda_j$ in the following way:

$$\omega|_{D_\lambda \cap \Lambda} = \begin{cases} 0 & \text{if } D_\lambda \cap \Lambda_j = \emptyset \\ \omega(\alpha) & \text{if } D_\lambda \cap \Lambda_j = \{\alpha\}. \end{cases}$$

We verify by induction that the extended sequence is in $X^{k-1}(\Lambda)$ for all $k \leq n$. It is clear that it belongs to $X^0(\Lambda)$.

Assume that $\omega \in X^{k-2}(\Lambda)$, $k \geq 2$, and consider $(\alpha_1, \dots, \alpha_k) \in \Lambda^k$. If all the points are in the same D_λ then $\Delta^{k-1}\omega(\alpha_1, \dots, \alpha_k) = 0$, so we may assume that $\alpha_1 \in D_\lambda$ and $\alpha_k \in D_{\lambda'}$ with $\lambda \neq \lambda'$. Then we have, for some $H_0 \in \text{Har}_+(\mathbb{D})$,

$$\rho(\alpha_1, \alpha_k) \geq e^{-\beta H_0(\alpha_1)}, \quad k \neq 1.$$

With this and the induction hypothesis it is clear that for some $H \in \text{Har}_+(\mathbb{D})$,

$$\begin{aligned} |\Delta^{k-1}\omega(\alpha_1, \dots, \alpha_k)| &= \left| \frac{\Delta^{k-2}\omega(\alpha_2, \dots, \alpha_k) - \Delta^{k-2}\omega(\alpha_1, \dots, \alpha_{k-1})}{b_{\alpha_1}(\alpha_k)} \right| \\ &\leq e^{\beta H_0(\alpha_1)} (e^{H(\alpha_2) + \dots + H(\alpha_k)} + e^{H(\alpha_1) + \dots + H(\alpha_{k-1})}). \end{aligned}$$

Taking for instance $\tilde{H} = H + \beta H_0 + \log 2$ we get

$$|\Delta^{k-1}\omega(\alpha_1, \dots, \alpha_k)| \leq e^{\tilde{H}(\alpha_1) + \dots + \tilde{H}(\alpha_k)},$$

thus $\omega(\Lambda) \in X^{k-1}(\Lambda)$. By assumption there exist $f \in \mathcal{N}$ interpolating the values $\omega(\Lambda)$. In particular f interpolates $\omega(\Lambda_j)$.

4. PROOF OF THE MAIN THEOREM. SUFFICIENCY

Assume $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_n$, where $\Lambda_j \in \text{Int } \mathcal{N}$, $j = 1, \dots, n$, and denote $\Lambda_j = \{\lambda_k^{(j)}\}_{k \in \mathbb{N}}$. Denote also by B_j the Blaschke product with zeros on Λ_j . We will use the following property of the Nevanlinna interpolating sequences (see Theorem 1.2 in [3]).

Lemma 4.1. *Let $\Lambda \in \text{Int } \mathcal{N}$ and let B the Blaschke product associated to Λ . There exists $H_1 \in \text{Har}_+(\mathbb{D})$ such that*

$$|B(z)| \geq e^{-H_1(z)} \rho(z, \Lambda) \quad z \in \mathbb{D}.$$

According to Proposition 2.1 we only need to see that $X^{n-1}(\Lambda) \subset \mathcal{N}|\Lambda$. Let then $\omega(\Lambda) \in X^{n-1}(\Lambda)$ and split it

$$\{\omega(\lambda)\}_{\lambda \in \Lambda} = \{\omega_k^{(1)}\}_{k \in \mathbb{N}} \cup \dots \cup \{\omega_k^{(n)}\}_{k \in \mathbb{N}},$$

where $\omega_k^{(j)} = \omega(\lambda_k^{(j)})$, $j = 1, \dots, n$, $k \in \mathbb{N}$. By Lemma 1.2 and the hypothesis $\{\omega_k^{(1)}\}_{k \in \mathbb{N}} \in X^0(\Lambda_1)$, hence there exists $f_1 \in \mathcal{N}$ such that

$$f_1(\lambda_k^{(1)}) = \omega_k^{(1)}, \quad k \in \mathbb{N}.$$

In order to interpolate also the values $\{\omega_k^{(2)}\}_k$ consider functions of the form

$$f_2(z) = f_1(z) + B_1(z)g_2(z).$$

Immediately $f_2(\lambda_k^{(1)}) = f_1(\lambda_k^{(1)}) = \omega_k^{(1)}$, $k \in \mathbb{N}$, and we will have $f_2(\lambda_k^{(2)}) = \omega_k^{(2)}$ as soon as we find $g_2 \in \mathcal{N}$ such that

$$g_2(\lambda_k^{(2)}) = \frac{\omega_k^{(2)} - f_1(\lambda_k^{(2)})}{B_1(\lambda_k^{(2)})}, \quad k \in \mathcal{N}.$$

Since $\Lambda_2 \in \text{Int } \mathcal{N}$ such g_2 will exist as soon as the sequence in the right hand side is majorized by a sequence of the form $\{e^{H(\lambda_k^{(2)})}\}_k$.

Given $\lambda_k^{(2)} \in \Lambda_2$ pick $\lambda_k^{(1)}$ such that $\rho(\lambda_k^{(2)}, \Lambda_1) = \rho(\lambda_k^{(2)}, \lambda_k^{(1)})$. There is no restriction in assuming that $\rho(\lambda_k^{(2)}, \lambda_k^{(1)}) \leq 1/2$. Then, by Lemma 4.1 there exists $H_1 \in \text{Har}_+(\mathbb{D})$ such that

$$|B_1(\lambda_k^{(2)})| \geq e^{-H_1(\lambda_k^{(2)})} \rho(\lambda_k^{(1)}, \lambda_k^{(2)}) \quad k \in \mathbb{N}.$$

Now, since $f_1(\lambda_k^{(1)}) = \omega_k^{(1)}$ we have

$$\begin{aligned} \left| \frac{\omega_k^{(2)} - f_1(\lambda_k^{(2)})}{B_1(\lambda_k^{(2)})} \right| &\leq \left| \frac{\omega_k^{(2)} - \omega_k^{(1)}}{B_1(\lambda_k^{(2)})} \right| + \left| \frac{f_1(\lambda_k^{(1)}) - f_1(\lambda_k^{(2)})}{B_1(\lambda_k^{(2)})} \right| \\ &\leq \left(\Delta^1(\omega_k^{(1)}, \omega_k^{(2)}) + \Delta^1(f_1(\lambda_k^{(1)}), f_1(\lambda_k^{(2)})) \right) e^{H_1(\lambda_k^{(2)})}. \end{aligned}$$

By hypothesis, and since $f_1 \in \mathcal{N}$, there exists $H_2 \in \text{Har}_+(\mathbb{D})$ such that

$$\Delta^1(\omega_k^{(1)}, \omega_k^{(2)}) + \Delta^1(f_1(\lambda_k^{(1)}), f_1(\lambda_k^{(2)})) \leq e^{H_2(\lambda_k^{(1)}) + H_2(\lambda_k^{(2)})},$$

and therefore, by Harnack's inequalities,

$$\left| \frac{\omega_k^{(2)} - f_1(\lambda_k^{(2)})}{B_1(\lambda_k^{(2)})} \right| \leq e^{H_2(\lambda_k^{(1)}) + H_2(\lambda_k^{(2)})} e^{H_1(\lambda_k^{(2)})} \leq e^{3(H_1 + H_2)(\lambda_k^{(2)})},$$

In general, assume that we have $f_{n-1} \in \mathcal{N}$ such that

$$f_{n-1}(\lambda_k^{(j)}) = \omega_k^{(j)} \quad k \in \mathbb{N}, j = 1, \dots, n-1.$$

We look for a function $f_n \in \mathcal{N}$ interpolating the whole Λ of the form

$$f_n = f_{n-1} + B_1 \cdots B_{n-1} g_n.$$

We need then $g_n \in \mathcal{N}$ with

$$g_n(\lambda_k^{(n)}) = \frac{\omega_k^{(n)} - f_{n-1}(\lambda_k^{(n)})}{B_1(\lambda_k^{(n)}) \cdots B_{n-1}(\lambda_k^{(n)})}, \quad k \in \mathbb{N}.$$

Let us see that the sequence of values in the right hand side of this identity have a majorant of the form $\{e^{H(\lambda_k^{(n)})}\}_k$.

Pick $\lambda_k^{(j)} \in \Lambda_j$, $j = 1, \dots, n-1$ such that $\rho(\lambda_k^{(n)}, \Lambda_j) = \rho(\lambda_k^{(n)}, \lambda_k^{(j)})$. There is no restriction in assuming that $\rho(\lambda_k^{(n)}, \lambda_k^{(j)}) \leq 1/2$. Since $f_{n-1}(\lambda_k^{(j)}) = \omega_k^{(j)}$, $j = 1, \dots, n-1$, an immediate computation shows that

$$\begin{aligned} \omega_k^{(n)} - f_{n-1}(\lambda_k^{(n)}) &= \left[\Delta^{n-1}(\omega_k^{(1)}, \dots, \omega_k^{(n-1)}, \omega_k^{(n)}) - \right. \\ &\quad \left. - \Delta^{n-1}(f_{n-1}(\lambda_k^{(1)}), \dots, f_{n-1}(\lambda_k^{(n-1)}), f_{n-1}(\lambda_k^{(n)})) \right] b_{\lambda_k^{(1)}}(\lambda_k^{(n)}) \cdots b_{\lambda_k^{(n-1)}}(\lambda_k^{(n)}). \end{aligned}$$

Again by Lemma 4.1, there exists $H_1 \in \text{Har}_+(\mathbb{D})$ such that

$$|B_j(\lambda_k^{(n)})| \geq e^{-H_1(\lambda_k^{(n)})} \rho(\lambda_k^{(j)}, \lambda_k^{(n)}), \quad k \in \mathbb{N}, j = 1, \dots, n-1.$$

Hence, by hypothesis and the fact that $f_{n-1} \in \mathcal{N}$ there exists $H \in \text{Har}_+(\mathbb{D})$ such that

$$\begin{aligned} \left| \frac{\omega_k^{(n)} - f_{n-1}(\lambda_k^{(n)})}{B_1(\lambda_k^{(n)}) \cdots B_{n-1}(\lambda_k^{(n)})} \right| &\leq [|\Delta^{n-1}(\omega_k^{(1)}, \dots, \omega_k^{(n)})| + |\Delta^{n-1}(f_{n-1}(\lambda_k^{(1)}), \dots, f_{n-1}(\lambda_k^{(n)}))|] e^{(n-1)H_1(\lambda_k^{(n)})} \\ &\leq e^{H(\lambda_k^{(1)}) + \dots + H(\lambda_k^{(n-1)}) + H(\lambda_k^{(n)}) + (n-1)H_1(\lambda_k^{(n)})}. \end{aligned}$$

Finally, by Harnack's inequalities, this is bounded by $e^{2n(H(\lambda_k^{(n)}) + H_1(\lambda_k^{(n)}))}$.

REFERENCES

- [1] Bruna, J.; Nicolau, A.; Øyma, K. *A note on interpolation in the Hardy spaces of the unit disc*. Proc. Amer. Math. Soc. 124 (1996), no. 4, 1197–1204.
- [2] Hartmann, A. *Une approche de l'interpolation libre gnralise par la thorie des oprateurs et caractrisation des traces $H^p|\Lambda$* . (French) [An approach to generalized free interpolation using operator theory and characterization of the traces $H^p|\Lambda$.] J. Operator Theory 35 (1996), no. 2, 281–316.
- [3] Hartmann, A., Massaneda, X., Nicolau, A. *Finitely generated Ideals in the Nevanlinna class*. arXiv:1605.08160.

- [4] Hartmann, A., Massaneda, X., Nicolau, A., Thomas, P. *Interpolation in the Nevanlinna and Smirnov classes and harmonic majorants*. J. Funct. Anal. 217 (2004), no. 1, 1-37.
- [5] Massaneda, X., Ortega-Cerdà, J., Ounaïes, M. *Traces of Hörmander algebras on discrete sequences*. Analysis and Mathematical Physics. Birkhäuser Verlag (2009) 397–408.
- [6] Naftalevič, A.G., *On interpolation by functions of bounded characteristic (Russian)*, Vilniaus Valst. Univ. Mokslų Darbai. Mat. Fiz. Chem. Mokslų Ser. **5** (1956), 5–27.
- [7] Vasyunin, V. I. *Traces of bounded analytic functions on finite unions of Carleson sets (Russian)*. Investigations on linear operators and the theory of functions, XII. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **126** (1983), 31–34.
- [8] Vasyunin, V. I. *Characterization of finite unions of Carleson sets in terms of solvability of interpolation problems (Russian)*. Investigations on linear operators and the theory of functions, XIII. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **135** (1984), 31–35.

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